On the stability of visco-elastic liquids in heated plane Couette flow

By D. M. HERBERT
Department of Mathematics, University of Manchester

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The stability of thermally stratified plane Couette flow of visco-elastic liquids, with respect to disturbances of small amplitude, is considered. It is found that an initial state of finite elastic stress is necessary for elasticity to effect stability. The critical Rayleigh number of the flow is shown to be decreased for all non-zero rates-of-strain. This greater instability is due solely to the variations of the apparent viscosity with shear rate. In this way, the presence of elasticity can be said to have a destabilizing effect on the flow.

Introduction

The stability of flow of visco-elastic liquids to disturbances of small amplitude, although of considerable practical importance, appears to have received little attention. In principle, it is possible to adapt classical stability analyses in order to investigate the effect of elasticity on the stability of such flows as Couette flow, Poiseuille flow, thermally stratified, flow, etc. However, in general, the usefulness of such an approach is limited by the mathematical difficulties introduced by the generalized rheological equations of state needed to describe visco-elastic phenomena.

The following treatment is restricted to the simplest cases and is intended to show the broad effects of elasticity on stability. It is shown that, to the first order, stability cannot be affected by elasticity unless the liquid concerned is in an initial state of finite elastic stress. Thus, the critical Rayleigh number for the stability of a horizontal layer of still visco-elastic liquid, heated from below, is precisely the same as that for a similar layer of Newtonian liquid. A steady shear flow gives the simplest instance of a state of finite stress and so the present investigation is concerned with the consequences of a small disturbance to thermally stratified plane Couette flow.

A qualitative decrease in the level of stability is found for those visco-elastic liquids in which the first mode of instability is of the same form as occurs in Newtonian liquids. The source of this greater instability is the variation in the apparent viscosity.

For certain dilute polymer solutions, an estimate of the decrease in critical Rayleigh Number, at large rates-of-strain, shows the change in the level of stability to be about 30 %, for the least elastic of these solutions.

The rheological equations of state

Oldroyd (1950, 1958) considered rheological equations of state for idealized, incompressible, visco-elastic liquids whose behaviour, at small variable shear stresses, is characterized by just three constants: a coefficient of viscosity, η_0, and two relaxation times, λ_1 and λ_2 ($< \lambda_1$). At small rates-of-strain, the stress tensor, p_{ik}, and the rate-of-strain tensor,

$$e_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right), \tag{1}$$

of these liquids are related by the rheological equations

$$p_{ik} = p'_{ik} - p\delta_{ik}, \tag{2}$$

$$p'_{ik} + \lambda_1 dp'_{ik}/dt = 2\eta_0(e_{ik} + \lambda_2 de_{ik}/dt), \tag{3}$$

where v_i denotes the velocity vector, δ_{ik} is the metric tensor and p is an isotropic pressure.

Oldroyd considered those visco-elastic liquids whose behaviour is defined by (2) and, as a generalization of (3),

$$\begin{aligned} p'_{ik} + \lambda_1 Dp'_{ik}/Dt + \mu_0 p'_{jj} e_{ik} - \mu_1(p'_{ij} e_{jk} + p'_{jk} e_{ij}) + \nu_1 p'_{jl} e_{jl} \delta_{ik} \\ = 2\eta_0(e_{ik} + \lambda_2 De_{ik}/Dt - 2\mu_2 e_{ij} e_{jk} + \nu_2 e_{jl} e_{jl} \delta_{ik}), \end{aligned} \tag{4}$$

where μ_0, μ_1, μ_2, ν_1 and ν_2 are arbitrary scalar constants, each with the dimension of time. The usual summation convention holds with respect to repeated suffices.

The 'material' derivative, D/Dt, is a total derivative following a typical fluid element, taking into account the rotational, as well as the translational, motion of the element. It is defined, for any tensor b_{ik}, as

$$\frac{Db_{ik}}{Dt} \equiv \frac{\partial b_{ik}}{\partial t} + v_j \frac{\partial b_{ik}}{\partial x_j} + \omega_{ij} b_{jk} + \omega_{kj} b_{ij}, \tag{5}$$

where
$$\omega_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_i} - \frac{\partial v_i}{\partial x_k}\right)$$
is the vorticity tensor.

The most commonly observed properties of non-Newtonian liquids were shown by Oldroyd to be exhibited by the liquids defined by equations (2) and (4). These liquids have a variable apparent viscosity in simple shear which decreases with increasing rate-of-strain from a limiting value, η_0, at low rates-of-strain, to another limiting value η_1 ($< \eta_0$) at high rates-of-strain. They exhibit the positive Weissenberg climbing effect, and have a distribution of normal stresses, corresponding to an extra tension along the streamlines, together with axial symmetry in many types of steady shearing flow.

For these properties to be exhibited at all rates-of-strain, the following restrictions, on the constants occurring in (4), were shown to be necessary:

$$\sigma_1 > \sigma_2 \geqslant \tfrac{1}{9}\sigma_1 > 0, \tag{6}$$

$$\lambda_1/\lambda_2 \geqslant \sigma_1/\sigma_2, \tag{7}$$

$$\mu_1 = \lambda_1, \quad \mu_2 = \lambda_2, \tag{8}$$

where
$$\sigma_1 = \nu_1(\lambda_1 - \tfrac{3}{2}\mu_0) + \mu_0\lambda_1, \quad \sigma_2 = \nu_2(\lambda_1 - \tfrac{3}{2}\mu_0) + \mu_0\lambda_2.$$

Equations of the undisturbed state

Rectangular Cartesian co-ordinates are chosen with the liquid lying between two horizontal plates. The lower plate, in the plane $y = 0$, is at rest and is maintained at a higher temperature than the upper plate which is in the plane $y = h$, and has velocity $(\gamma h, 0, 0)$ where γ is constant.

The equations of steady motion may be written in the form

$$\frac{\partial p'_{ik}}{\partial x_k} = \frac{\partial p}{\partial x_i} + \rho g_i, \tag{9}$$

where $g_i = (0, g, 0)$. There is a solution of these equations in which the p'_{ik} are all constant and

$$\partial p/\partial y = -\rho g. \tag{10}$$

Under these conditions, equations (2) and (4) can be solved (Oldroyd 1958) to give the stresses required to maintain a steady shearing flow, namely:

$$\left.\begin{aligned} p_{11} &= \{(2\lambda_1 - \nu_1)F(\gamma) - (2\lambda_2 - \nu_2)\}\eta_0\gamma^2 - p,\\ p_{22} &= p_{33} = -\{\nu_1 F(\gamma) - \nu_2\}\eta_0\gamma^2 - p,\\ p_{12} &= p'_{12} = \eta_0\gamma F(\gamma), \; p_{23} = p'_{23} = 0, p_{13} = p'_{13} = 0, \end{aligned}\right\} \tag{11}$$

where $F(\gamma) = (1+\sigma_2\gamma^2)/(1+\sigma_1\gamma^2)$.

The form of the shear stress, p_{12}, is such that the liquid may be considered to have a single variable coefficient of viscosity, $\eta = \eta_0 F(\gamma)$.

The equation of continuity is identically satisfied in the undisturbed state while the energy equation reduces to

$$0 = k\,\partial^2 T/\partial y^2, \tag{12}$$

where T is the temperature and k the thermal diffusivity. From (12), it follows that $T = T_0 + \beta y$, where β is the temperature gradient. The suffix 0 refers to conditions at the lower plate. Using the equation of state

$$\rho = \rho_0\{1 - \alpha(T - T_0)\},$$

(10) becomes

$$\partial p/\partial y = -g\rho_0(1 - \alpha\beta y). \tag{13}$$

If a liquid is initially in a state of zero stress, a small-amplitude disturbance will give rise to a rate-of-strain tensor, e_{ik}, and a stress tensor, p'_{ik}, with components of the first order of smallness. Under these conditions, the linearized form of equation (4) is

$$p'_{ik} + \lambda_1\,\partial p'_{ik}/\partial t = 2\eta_0(e_{ik} + \lambda_2\,\partial e_{ik}/\partial t).$$

It is usual to assume, where it cannot be proved, that the equations governing neutral stability are given by taking all time variations to be zero, i.e. by considering states of steady secondary flow. This reduces the rheological equations of state to those for a Newtonian liquid. It follows that the introduction of elasticity can have no effect on stability unless there is an initial distribution of finite elastic stress.

The linearized equations

A small amplitude disturbance will create a velocity field of the form $(u+\gamma y, v, w)$, a temperature $T+\theta$ and a stress tensor

$$p_{ik} = P'_{ik} + p'_{ik} - (P+p)\,\delta_{ik}, \tag{14}$$

where P'_{ik} and P refer to the initial state and all the quantities u, v, w, θ, p'_{ik} and p are sufficiently small to permit the equations to be linearized.

In the absence of shear, the steady cellular pattern of flow, which occurs as first mode of instability for a general disturbance, is that of the familiar Bénard cells. The introduction of shear makes such a steady cellular motion no longer possible for a disturbance of general type. However, it was shown by Jeffreys (1928) that, in a Newtonian liquid, a steady cellular motion can still exist but only for disturbances that are independent of displacement in the direction of the mainstream flow, i.e. the x-direction. In this mode, the flow is composed of roll cells whose longitudinal axis is in the x-direction and about which fluid particles describe spiral paths. The only effect that the presence of shear has on the stability of flow of a Newtonian liquid was shown by Jeffreys to be that of stabilizing disturbances that are periodic in the x-direction, the critical Rayleigh number being unaffected.

It seems reasonable to suppose that, in a liquid which is slightly elastic, the same pattern of motion, as for a Newtonian liquid, will occur as the first mode of instability. In the liquids which are more strongly elastic, the first mode may well be considerably different in form, but such liquids fall outside the scope of the present work. It is assumed, then, that only those disturbances that are independent of displacement in the x-direction need be considered as it is disturbances of this type which give rise to the first mode of instability.

Hence, the equations governing the development of a small disturbance to the thermally stratified Couette flow are

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{15}$$

$$\frac{\partial u}{\partial t} + \gamma v = \frac{1}{\rho}\left(\frac{\partial p'_{12}}{\partial y} + \frac{\partial p'_{13}}{\partial z}\right), \tag{16}$$

$$\frac{\partial v}{\partial t} = -g - \frac{1}{\rho}\frac{\partial}{\partial y}(P+p) + \frac{1}{\rho}\left(\frac{\partial p'_{22}}{\partial y} + \frac{\partial p'_{23}}{\partial z}\right), \tag{17}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\rho}\left(\frac{\partial p'_{32}}{\partial y} + \frac{\partial p'_{33}}{\partial z}\right), \tag{18}$$

$$\frac{\partial \theta}{dt} + \beta v = k\nabla_1^2\theta, \tag{19}$$

where
$$\nabla_1^2 \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Using (13), equation (17) reduces to

$$\frac{\partial v}{\partial t} = \alpha g\theta - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\left(\frac{\partial p'_{22}}{\partial y} + \frac{\partial p'_{23}}{\partial z}\right), \tag{20}$$

and ρ may now be regarded as constant. Eliminating p between equations (18) and (20) gives

$$\frac{\partial}{\partial t}\left(\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial z}\right) = -\alpha g\frac{\partial \theta}{\partial z} + \frac{1}{\rho}\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)p'_{23} + \frac{1}{\rho}\frac{\partial^2}{\partial y\,\partial z}(p'_{33} - p'_{22}). \tag{21}$$

The equations for neutral stability

As is customary in stability analyses, a disturbance is Fourier—analysed into its harmonic modes and the wave-numbers considered separately. As independence of x has been assumed already, the solutions of the linearized equations are taken to be of the form:

$$\left.\begin{aligned} p'_{jk}(y,z;\,t) &= p'_{jk}(y)\exp(\Omega t + i\kappa z),\\ \theta(y,z;\,t) &= \theta(y)\exp(\Omega t + i\kappa z),\\ u(y,z;\,t) &= u(y)\exp(\Omega t + i\kappa z), \end{aligned}\right\} \tag{22}$$

and similarly for v and w.

Equations (15), (16), (19) and (21) become, on using (22):

$$dv/dy + i\kappa w = 0, \tag{23}$$

$$\Omega u + \gamma v = \frac{1}{\rho}\left(\frac{dp'_{12}}{dy} + i\kappa p'_{13}\right), \tag{24}$$

$$-\beta v = (\Omega - k\nabla^2)\theta, \tag{25}$$

$$\frac{\Omega}{\kappa^2}\nabla^2 v = -\alpha g\theta + \frac{1}{i\kappa\rho}\left(\frac{d^2}{dy^2} + \kappa^2\right)p'_{23} + \frac{1}{\rho}\frac{d}{dy}(p'_{33} - p'_{22}), \tag{26}$$

where $\nabla^2 \equiv d^2/dy^2 - \kappa^2$ and equation (23) has been used in equation (26).

The linearized rheological equations for the stresses occurring in equation (26) are

$$(1 + \lambda_1\Omega)p'_{23} - (i\eta_0/\kappa)\{F(\gamma) + \lambda_2\Omega\}(d^2/dy^2 + \kappa^2)v = 0,$$
$$(1 + \lambda_1\Omega)p'_{22} + \nu_1\gamma p'_{12} - 2\eta_0\{F(\gamma) - \lambda_2\Omega\}dv/dy + \eta_0\gamma\{\nu_1 F(\gamma) - 2\nu_2\}du/dy = 0,$$
$$(1 + \lambda_1\Omega)p'_{33} + \nu_1\gamma p'_{12} + 2\eta_0\{F(\gamma) + \lambda_2\Omega\}dv/dy + \eta_0\gamma\{\nu_1 F(\gamma) - 2\nu_2\}du/dy = 0.$$

Hence

$$p'_{23} = \frac{i\eta_0}{k}\left(\frac{F(\gamma) + \lambda_2\Omega}{1 + \lambda_1\Omega}\right)\left(\frac{d^2}{dy^2} + \kappa^2\right)V, \tag{27}$$

$$p'_{33} - p'_{22} = -\frac{4\eta_0 F(\gamma)}{1 + \lambda_1\Omega}\frac{dv}{dy}. \tag{28}$$

In their treatment of the classical problem, Pellew & Southwell (1940) were able to prove that neutral stability conditions arise from steady disturbances. The time-dependent nature of the stresses in the above expressions appear to prevent their proof from being adapted to the present equations. It has to be assumed, therefore, that the equations for neutral stability are given when the disturbances are restricted to be steady, i.e. $\Omega = 0$.

Substituting the simplified forms of (27) and (28) into equation (26) and then eliminating θ by means of equation (25) gives

$$\{k\nu F(\gamma)\,\nabla^6 - \kappa^2\alpha\beta g\}\,v = 0, \tag{29}$$

where $\nu = \eta_0/\rho$, is the kinematic viscosity. A substitution of $y = h\zeta$ into equation (29) gives the non-dimensional form of the stability equation for a visco-elastic liquid

$$(D^2 - a^2)^3\,v = -\{Ra\,a^2/F(\gamma)\}\,v, \tag{30}$$

where $D = d/d\zeta$, $a = \kappa h$, and $Ra = -\alpha\beta g h^4/k\nu$ is the Rayleigh number based on η_0.

For a given value of γ, solutions of the stability equation involve just the two parameters, Ra and a. This latter parameter is characteristic only of the size of the cells of the convective motion: the shape of the cells is not specified. As there are no fixed vertical boundaries in the flow, the range of values that a can take is unrestricted. The cell pattern that occurs is the one for which the corresponding Rayleigh number has a minimum value. This critical value, $(Ra)_{\text{crit}}$ is the criterion of thermal stability of the flow in that, if it is exceeded, a steady cellular pattern of flow will develop. The value of a for this particular cell pattern corresponds to the critical Rayleigh number.

It is, perhaps, worthwhile noting, at this stage, that non-Newtonian inelastic liquids are covered by the above analysis. For example, the Reiner–Rivlin model

$$p'_{ik} = \alpha_1 e_{ik} + \alpha_2 e_{ij} e_{jk},$$

where α_1 and α_2 are arbitrary functions of the principal invariants of the matrix e_{ij}, leads essentially to the same equations and results as the Oldroyd fluid model considered above, for which an apparent viscosity is the only parameter introduced by the non-Newtonian nature of the flow.

It is known (Pellew & Southwell 1940) that the least value of the Rayleigh number for which a solution of the classical stability equation exists is 1708, corresponding to $a = 3{\cdot}13$. The critical Rayleigh number for the flow of a visco-elastic liquid may be written therefore in the form

$$(Ra)_{\text{crit}} = 1708F(\gamma). \tag{31}$$

As $F(\gamma)$ is a decreasing function of γ, it follows at once that the critical Rayleigh number is smaller for a visco-elastic liquid than for a Newtonian liquid, at any non-zero rate-of-strain.

Numerical values for σ_1 and σ_2 are needed in order to find the quantitative decrease in $(Ra)_{\text{crit}}$ as the rate-of-strain is increased. Of the constants involved in these parameters, values for λ_1 and λ_2 only have been deduced from experimental results. However, from observations that have been made on dilute polymer solutions by Thoms & Strawbridge (1953), it appears that the decrease in the apparent viscosity is considerably greater at large rates-of-strain than that predicted by Oldroyd's theory. Thus, the theoretical decrease in $F(\gamma)$, at large rates-of-strain, provides a conservative estimate of the actual decrease in the critical Rayleigh number. Now, for large rates-of-strain $F(\gamma) \sim \sigma_2/\sigma_1$, and from condition (7), $\sigma_2/\sigma_1 = \lambda_2/\lambda_1$. This gives, as a simplified form of (31),

$$(Ra)_{\text{crit}} = 1708\lambda_2/\lambda_1. \tag{32}$$

For the dilute polymer solutions mentioned above, the ratio λ_2/λ_1 decreased from 2/3 to 3/13 as the polymer concentration was increased. Thus, even on this conservative estimate, the decrease in the critical Rayleigh number for the most weakly elastic polymer solution is about 30 %.

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